Full Slepian-Bangs Formula for Fisher Information on Lie Groups

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***Abstract*—In this communication we develop a new full Slepian-Bangs formula adapted to observations on Lie Groups (LGs) distributed according to a Gaussian distribution on LGs (CGD), where both the LG mean and LG covariance depend on an unknown LG parameter. This formulation, which can be seen as a generalization on LGs of the full Slepian-Bangs formula for the Euclidean Gaussian distribution, is obtained using LG tools and properties of the CGD. A closed-form expression is then obtained for a modified Wahba's problem where the observations' covariance matrix depends on the unknown rotation matrix. Such expression is validated through numerical simulations.**

***Index Terms*—Slepian-Bangs formula, Gaussian distribution on Lie groups, Fisher information.**

I. Introduction

The Fisher information is a central concept which appears in several fields of science. It is primarily known in the asymptotic statistical theory [1] (maximum likelihood estimation, hypothesis testing, etc.), but it is also well investigated in classical and quantum physics [2]. One can also mention the Fisher-Rao distance [3], based on the Fisher information, that serves as a cornerstone in information geometry [4]. In the signal processing community, the Fisher information is primarily used through the Cramér-Rao bound (CRB). This bound allows to assess the possible achievable performance limits in terms of variance or mean square error, for an unbiased estimator, in the parametric context [5] but also, more recently, in the semi-parametric [6] or even in the non-parametric contexts [7]. As a byproduct, closed-form expressions of the CRB for a particular problem have been used in engineering for system design [8].

Particularly, in the context of parametric estimation, the success of the CRB can be attributed to the so-called Slepian-Bangs formula [9], [10], which provides a closed-form expression (i.e., free from expectation operators) of the Fisher information for the fundamental case where the observations $\mathbf{z} \in \mathbb{R}^N$ are assumed to be Gaussian, with mean $\mathbf{m}$ and covariance matrix $\boldsymbol{\Sigma}$ parameterized by the unknown parameter vector $\boldsymbol{\theta} \in \mathbb{R}^P$; $\mathbf{z} \sim \mathcal{N}\left(\mathbf{m}\left(\boldsymbol{\theta}\right), \boldsymbol{\Sigma}\left(\boldsymbol{\theta}\right)\right)$. This formula encompasses a large class of engineering problems, and only the derivatives of $\mathbf{m}\left(\boldsymbol{\theta}\right)$ and $\boldsymbol{\Sigma}\left(\boldsymbol{\theta}\right)$ w.r.t. $\boldsymbol{\theta}$ have to be computed in order to obtain interesting results in terms of achievable performance limits or for system design. Since the seminal work of Slepian and Bangs [9], this formula has been extended in many ways. One can mention the extension to complex-valued observations and/or parameters [11], also the extension to a broader class of probability distributions of the observations [12], or the case where the covariance matrix is non-circular [13].

While these works focused on Euclidean observations and parameters (i.e., $\mathbf{z} \in \mathbb{R}^N$ and $\boldsymbol{\theta} \in \mathbb{R}^P$), recent estimation problems are more effectively characterized with observations and/or parameters lying in more complex/structured spaces. An important example is when the parameter to be estimated is a covariance matrix. The original Slepian-Bangs formula overlooks the fact that this parameter lies in the space of symmetric positive definite (SPD) matrices. Consequently, the obtained Fisher information (and CRB) is just an approximation. Thus, to address geometrical structures of the parameter space, non-Euclidean Fisher information matrices have been proposed. When the parameter lies in a general Riemannian manifold, the so-called intrinsic CRB has been developed and studied in [14] and, for instance, it has been shown that classical results known for covariance matrix estimation (bias, covariance error) in the Gaussian context are invalidated in light of the SPD matrix point of view.

In this work, we focus on Matrix Lie groups, a Riemmanian manifold equipped with a group structure, that have important applications in engineering such as computer vision [15], robotics [16] and GNSS navigation [17]. Previous works [18], [19], [20] derived intrinsic CRBs for LG (LG-CRBs). Furthermore, closed-form expressions of the LG Fisher information (LG-FIM) have been previously provided for some particular scenarios: when only the mean parameter (LG-mean) is unknown [21], [22], especially for Gaussian distributions on LGs (CGD). In this context, i.e., when both parameters and observations lie in a (possibly distinct) LG, the need

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for a Slepian-Bangs type formula of the Fisher information (devoid from the expectation) for CGDs is of utmost interest. A first simplified version has been proposed in [22], when the LG-parameter depends on the LG-mean with a known LG-covariance (LG-SP).

The main contribution of this paper is to extend the LG-SP [22] by considering the general case where the LG-covariance also depends on the LG-parameter. This general framework yields a full Slepian-Bangs formula on LGs (LG-F-SP). This situation is typically encountered when the unknown LG-parameter is correlated with the LG noise of observations. Examples include the Wahba's model [23] or the dispersion model of space debris [24] on the LG $SE(3)$. To achieve this, we use the theoretical expression of the LG-FIM for CGDs. Contrary to [22], the LG derivatives are computed according to the unknown parameter nested in the LG-covariance. This implies that the resulting Slepian-Bangs formula reveals correlation terms between LG-mean and LG-covariance. Then, a closed-form expression of the LG-F-SP is derived for a modified Wahba's problem in which the noise, depending on an unknown rotation matrix, lies on the LG $SO(3)$. By inverting the derived LG-F-SP, we obtain the LG-CRB for this problem. Finally, the validity of these expressions is assessed through numerical simulations.

II. Reviewing the Slepian-Bangs on LGs

A. *LG definition*

A matrix LG $G \subset \mathbb{R}^{n\times n}$ is a matrix space that respects the properties of smooth manifold and group. This implies the definition of a tangent space at the identity, also referred to as the Lie algebra, and denoted $\mathfrak{g}$, directly connected to the tangent space at each point $\mathbf{X} \in G$ by the group operation. The exponential and logarithmic maps, denoted respectively, $\mathrm{Exp}_G : \mathfrak{g} \to G$ and $\mathrm{Log}_G : G \to \mathfrak{g}$ associate each element of the LG to $\mathfrak{g}$. Since the latter is isomorphic to $\mathbb{R}^m$, we can define two bijections $[.]^\wedge : \mathbb{R}^m \to \mathfrak{g}$ and $[.]^\vee : \mathfrak{g} \to \mathbb{R}^m$. Then, the exponential and logarithmic mappings can be reformulated, $\forall\, \mathbf{a} \in \mathbb{R}^m$, $\mathrm{Exp}^\wedge_G(\mathbf{a}) = \mathrm{Exp}_G([\mathbf{a}]^\wedge_G)$ and $\forall\, \mathbf{X} \in G, [\mathrm{Log}_G(\mathbf{X})]^\vee_G = \mathrm{Log}^\vee_G(\mathbf{X})$. For more details on LG background, readers can refer to [21], [22].

B. *Simplified Slepian-Bangs formula on LGs*

Let us consider a set of independent observations denoted as $\mathbf{Z} = \{\mathbf{Z}_1, \ldots, \mathbf{Z}_N\}$, lying on a LG G', following a CGD distribution, as described in [25],

$$\mathbf{Z}_i = \mathbf{H}_i(\mathbf{M})\,\mathrm{Exp}^\wedge_{G'}(\mathbf{n}_i) \quad \mathbf{n}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}) \;\; \forall i \in \{1, \ldots, N\}, \quad (1)$$

where $\mathbf{M}$ is the unknown parameter belonging to another LG G and $\mathbf{H}_i : G \to G'$ is a smooth function. The Intrinsic Mean Squared Error (IMSE) between an unbiased estimator $\widehat{\mathbf{M}}$ and $\mathbf{M}$, in the intrinsic sense as defined in [22], is given by

$$\mathbf{E} = \mathbb{E}\left[\mathrm{Log}^\vee_G\left(\mathbf{M}^{-1}\widehat{\mathbf{M}}\right)\,\mathrm{Log}^\vee_G\left(\mathbf{M}^{-1}\widehat{\mathbf{M}}\right)^\top\right]. \quad (2)$$

This error is lower-bounded by the CRB on LGs (LG-CRB), which is the inverse of the Slepian-Bangs formula $\mathcal{I}$ (LG-SP):

$$\mathbf{P}_{ICRB} = \mathcal{I}^{-1}, \quad (3)$$

$$\mathcal{I} = \sum_{i=1}^{N} \mathcal{L}^R_{\mathbf{H}_i(\mathbf{M})}\left(\mathbb{E}\left[\tilde{\boldsymbol{\psi}}_i^\top \boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\psi}}_i\right]\right)\left(\mathcal{L}^R_{\mathbf{H}_i(\mathbf{M})}\right)^\top, \quad (4)$$

where $\tilde{\boldsymbol{\psi}}_i = \boldsymbol{\psi}_{G'}(\mathrm{Log}^\vee_{G'}\left(\mathbf{H}_i(\mathbf{M})^{-1}\,\mathbf{Z}_i\right)\,)$ is the inverse of the left Jacobian of G', and $\mathcal{L}^R_{\mathbf{H}_i(\mathbf{M})}$ represents the right Lie derivative of $\mathbf{H}_i$. For further details regarding the computation of these quantities, readers can refer to [24].

III. Full Slepian-Bangs on Lie groups: Formulation and Demonstration

In this section, we present and demonstrate the full Slepian-Bangs (F-LG-SP), which extends (4) to the general case where the covariance matrix of (1) depends on $\mathbf{M}$.

A. *Full Slepian-Bangs formula on LGs*

We consider now that the LG observation $\mathbf{Z}_i \in G'$ (of dimension S', $i \in \{1, \ldots, N\}$) is related to $\mathbf{M} \in G$ (of dimension S), through the concentrated Gaussian model:

$$\mathbf{Z}_i = \mathbf{H}_i(\mathbf{M})\,\mathrm{Exp}^\wedge_{G'}(\boldsymbol{\epsilon}_i)\;,\;\; \boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}(\mathbf{M})). \quad (5)$$

*Theorem III-A.1 (**LG-F-SP for a CGD**): the LG-F-SP $\mathcal{I}$ on* $\mathbf{M}$ *for the observation model* (5) *is given by:*

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_2^\top + \mathcal{I}_3 \quad (6)$$

wherein,

$$\mathcal{I}_1 = \sum_{i=1}^{N} \mathcal{L}^\top_{\mathbf{H}_i(\mathbf{M})}\boldsymbol{\psi}_i^\top \boldsymbol{\Sigma}(\mathbf{M})^{-1}\boldsymbol{\psi}_i\,\mathcal{L}_{\mathbf{H}_i(\mathbf{M})} \quad (7)$$

$$\begin{aligned}\mathcal{I}_2 =\;& \frac{1}{2}\,\mathbb{E}\left(\mathcal{L}^\top_{\mathbf{H}_i(\mathbf{M})}\boldsymbol{\psi}_i^\top\;\boldsymbol{\Sigma}^{-1}\boldsymbol{l}_i\;\left\{\boldsymbol{l}_i^\top \mathrm{d}\boldsymbol{\Sigma}_1^{-1}\,\boldsymbol{l}_i, \ldots, \boldsymbol{l}_i^\top \mathrm{d}\boldsymbol{\Sigma}_S^{-1}\boldsymbol{l}_i\right\}\right)\\ &+ \frac{1}{2}\mathbb{E}\left(\mathcal{L}^\top_{\mathbf{H}_i(\mathbf{M})}\boldsymbol{\psi}_i^\top\;\boldsymbol{\Sigma}^{-1}\boldsymbol{l}_i\right)\mathrm{dlog}|\boldsymbol{\Sigma}|^\top \end{aligned} \quad (8)$$

$$(\mathcal{I}_3)_{k,l} = \frac{N}{2}\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\mathrm{d}\boldsymbol{\Sigma}_k\,\boldsymbol{\Sigma}^{-1}\mathrm{d}\boldsymbol{\Sigma}_l\right)\forall (k,l) \in [\![1, \ldots, S]\!]^2 \quad (9)$$

where we make use of the following notations:

$$\begin{aligned}
\mathcal{L}_{\mathbf{H}_i(\mathbf{M})} &= \left.\frac{\partial\, \boldsymbol{l}_{G'}(\mathbf{H}_i(\mathbf{M}\,\mathrm{Exp}^\wedge_G(\boldsymbol{\delta})\,), \mathbf{Z}_i)}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}} \in \mathbb{R}^{S'\times S},\\
\boldsymbol{\psi}_i &= \boldsymbol{\psi}_{G'}(\mathrm{Log}^\vee_{G'}\left(\mathbf{H}_i(\mathbf{M})^{-1}\mathbf{Z}_i\right)\,) \in \mathbb{R}^{S'\times S'},\\
\boldsymbol{l}_i &= \mathrm{Log}^\vee_{G'}\left(\mathbf{H}_i(\mathbf{M})^{-1}\mathbf{Z}_i\right) \;\in \mathbb{R}^{S'},\\
\boldsymbol{\Sigma} &= \boldsymbol{\Sigma}(\mathbf{M}) \in \mathbb{R}^{S'\times S'},\\
\mathrm{d}\boldsymbol{\Sigma}_l &= \left.\frac{\partial \boldsymbol{\Sigma}(\mathbf{M}\,\mathrm{Exp}^\wedge_G(\boldsymbol{\delta})\,)}{\partial \delta_l}\right|_{\boldsymbol{\delta}=\mathbf{0}} \in \mathbb{R}^{S'\times S'}\forall l \in [\![1, \ldots, S]\!]
\end{aligned}$$

and $\mathrm{dlog}|\boldsymbol{\Sigma}| = \left.\frac{\partial \mathrm{log}|\boldsymbol{\Sigma}(\mathbf{M}\,\mathrm{Exp}^\wedge_G(\boldsymbol{\delta})\,)|}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}} \in \mathbb{R}^S$.

In the following, for the sake of clarity, we define:

$$\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma},\mathbf{M}}(\mathbf{Z}, \mathbf{M}) \triangleq \left.\frac{\partial \mathrm{log}\,\mathcal{N}(\mathbf{Z}; \mathbf{H}_i(\mathbf{M}\,\mathrm{Exp}^\wedge_G(\boldsymbol{\delta}))\;, \boldsymbol{\Sigma}(\mathbf{M}\,\mathrm{Exp}^\wedge_G(\boldsymbol{\delta})))}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}$$

$$\mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z}, \mathbf{M}) \triangleq \left.\frac{\partial \mathrm{log}\,\mathcal{N}(\mathbf{Z}; \mathbf{H}_i(\mathbf{M}\,\mathrm{Exp}^\wedge_G(\boldsymbol{\delta}))\;, \boldsymbol{\Sigma}(\mathbf{M}))}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}$$

$$\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma}}(\mathbf{Z}, \mathbf{M}) \triangleq \left.\frac{\partial\, \mathrm{log}\,\mathcal{N}(\mathbf{Z}; \mathbf{H}_i(\mathbf{M}), \boldsymbol{\Sigma}(\mathbf{M}\,\mathrm{Exp}^\wedge_G(\boldsymbol{\delta}))\,)}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}$$

B. Demonstration

By definition, the LG-FIM is given by [21, eq. (11)]:

$$\mathcal{I} = \mathbb{E}\left(\frac{\partial lp(\mathbf{M},\boldsymbol{\delta}_1)}{\partial \boldsymbol{\delta}_1}\bigg|_{\boldsymbol{\delta}_1=\mathbf{0}} \frac{\partial lp(\mathbf{M},\boldsymbol{\delta}_2)}{\partial \boldsymbol{\delta}_2}^\top\bigg|_{\boldsymbol{\delta}_2=\mathbf{0}} \right), \quad (10)$$

where $lp(\mathbf{M},\boldsymbol{\delta}) \triangleq \log p(\mathbf{Z}|\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})\,)$ and can be written, by independence of each $\mathbf{Z}_i$, as

$$\sum_{i=1}^N \mathbb{E}(\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma},\mathbf{M}}(\mathbf{Z}_i,\mathbf{M})\,\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma},\mathbf{M}}(\mathbf{Z}_i,\mathbf{M})^\top). \quad (11)$$

Furthermore, the generic term can be divided in the following way: $\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma},\mathbf{M}}(\mathbf{Z},\mathbf{M}) = \mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z},\mathbf{M}) + \mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma}}(\mathbf{Z},\mathbf{M})$. Then, we have

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_2^\top + \mathcal{I}_3$$

with $\mathcal{I}_1 = \sum_{i=1}^N \mathbb{E}\left(\mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z}_i,\mathbf{M})\,\mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z}_i,\mathbf{M})^\top\right)$ $\mathcal{I}_2 = \sum_{i=1}^N \mathbb{E}\left(\mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z}_i,\mathbf{M})\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma}}(\mathbf{Z}_i,\mathbf{M})^\top\right)$ and $\mathcal{I}_3 = \sum_{i=1}^N \mathbb{E}\left(\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma}}(\mathbf{Z}_i,\mathbf{M})\,\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma}}(\mathbf{Z}_i,\mathbf{M})^\top\right)$.

Now, we are interested in detailing $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$.

1) Computation of $\mathcal{I}_1$:

First, it is straightforward to see that:

$$\mathbb{E}\left(\mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z}_i,\mathbf{M})\,\mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z}_i,\mathbf{M})^\top\right) = - \mathbb{E}\left(\frac{\partial \log \mathcal{N}(\mathbf{Z}_i;\mathbf{H}_i(\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\;\;\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\,),\boldsymbol{\Sigma}(\mathbf{M}))}{\partial \boldsymbol{\delta}_1\,\partial \boldsymbol{\delta}_2}\bigg|_{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2=\mathbf{0}} \right) \quad (12)$$

Second, it is proved in [22, Theorem 4.2.1] that the previous expression is equal to $\mathcal{L}_{\mathbf{H}_i(\mathbf{M})}^\top \boldsymbol{\psi}_i^\top \boldsymbol{\Sigma}(\mathbf{M})^{-1}\,\boldsymbol{\psi}_i\,\mathcal{L}_{\mathbf{H}_i(\mathbf{M})}$.

2) Computation of $\mathcal{I}_2$:

To compute $\mathcal{I}_2$, we have to determine an expression of $\mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z}_i,\mathbf{M})$ and $\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma}}(\mathbf{Z}_i,\mathbf{M})$. First, we know that up to a constant value $K \in \mathbb{R}$:

$$\log \mathcal{N}(\mathbf{Z}_i;\mathbf{H}_i(\mathbf{M}),\boldsymbol{\Sigma}(\mathbf{M})) = K - \log|\boldsymbol{\Sigma}| - \frac{1}{2}\|\boldsymbol{l}_i\|_{\boldsymbol{\Sigma}}^2, \quad (13)$$

which implies that, by usual derivations:

$$\mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z}_i,\mathbf{M}) = -\mathrm{d}\boldsymbol{l}_i \boldsymbol{\Sigma}^{-1}\boldsymbol{l}_i \quad (14)$$

with $\mathrm{d}\boldsymbol{l}_i = \frac{\partial \boldsymbol{l}_i^{\boldsymbol{\delta}}}{\partial \boldsymbol{\delta}}\Big|_{\boldsymbol{\delta}=\mathbf{0}}$ and $\boldsymbol{l}_i^{\boldsymbol{\delta}} = \mathrm{Log}_{G'}^\vee\left(\mathbf{H}_i(\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})\,)^{-1}\mathbf{Z}_i\right)$.

Furthermore, the BCH formula [24, eq. (2)], informs us that $\boldsymbol{l}_i^{\boldsymbol{\delta}} = \boldsymbol{l}_i - \mathcal{L}_{\mathbf{H}_i}(\mathbf{M})^\top \boldsymbol{\psi}_i^\top\,\boldsymbol{\delta}$ then, by deriving according to $\boldsymbol{\delta}$,

$$\mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z}_i,\mathbf{M}) = -\mathcal{L}_{\mathbf{H}_i(\mathbf{M})}^\top \boldsymbol{\psi}_i^\top\,\boldsymbol{\Sigma}^{-1}\boldsymbol{l}_i \quad (15)$$

On the other hand, by derivation of the equation (13) according to $\boldsymbol{\Sigma}(\mathbf{M})$, we end up with:

$$\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma}}(\mathbf{Z}_i,\mathbf{M}) = -\{\boldsymbol{l}_i^\top \mathrm{d}\boldsymbol{\Sigma}_1^{-1}\,\boldsymbol{l}_i,\ldots,\boldsymbol{l}_i^\top \mathrm{d}\boldsymbol{\Sigma}_S^{-1}\boldsymbol{l}_i\} - \mathrm{dlog}|\boldsymbol{\Sigma}| \quad (16)$$

Consequently,

$$\mathbb{E}(\mathrm{d}\mathcal{LN}_{\mathbf{M}}(\mathbf{Z}_i,\mathbf{M})\,\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma}}(\mathbf{Z}_i,\mathbf{M})^\top) = \frac{1}{2}\sum_{i=1}^N \mathbb{E}\left(\mathcal{L}_{\mathbf{H}_i(\mathbf{M})}^\top \boldsymbol{\psi}_i^\top\,\boldsymbol{\Sigma}^{-1}\boldsymbol{l}_i\,\left\{\boldsymbol{l}_i^\top \mathrm{d}\boldsymbol{\Sigma}_1^{-1}\,\boldsymbol{l}_i,\ldots,\boldsymbol{l}_i^\top \mathrm{d}\boldsymbol{\Sigma}_S^{-1}\boldsymbol{l}_i\right\}\right) + \frac{1}{2}\sum_{i=1}^N \mathbb{E}\left(\mathcal{L}_{\mathbf{H}_i}(\mathbf{M})^\top \boldsymbol{\psi}_i^\top\,\boldsymbol{\Sigma}^{-1}\boldsymbol{l}_i\right)\mathrm{dlog}|\boldsymbol{\Sigma}| \quad (17)$$

3) Computation of $\mathcal{I}_3$:

We recall that,

$$\mathcal{I}_3 = \sum_{i=1}^N \mathbb{E}\left(\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma}}(\mathbf{Z}_i,\mathbf{M})\,\mathrm{d}\mathcal{LN}_{\boldsymbol{\Sigma}}(\mathbf{Z}_i,\mathbf{M})^\top\right). \quad (18)$$

By taking advantage of (16), we can develop $\mathcal{I}_3$ into four terms:

$$\begin{aligned}\mathcal{I}_3 &= \frac{N}{4}\mathbb{E}\left(\mathrm{dlog}|\boldsymbol{\Sigma}|\,\mathrm{dlog}|\boldsymbol{\Sigma}|^\top\right) \\ &+ \frac{1}{4}\sum_{i=1}^N \mathbb{E}\left(\mathrm{dlog}|\boldsymbol{\Sigma}|\,\left\{\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{l}_i,\ldots,\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_S^{-1}\boldsymbol{l}_i\right\}\right) \\ &+ \frac{1}{4}\sum_{i=1}^N \mathbb{E}\left(\left\{\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{l}_i,\ldots,\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_S^{-1}\boldsymbol{l}_i\right\}^\top\,\mathrm{dlog}|\boldsymbol{\Sigma}|^\top\right) \\ &+ \frac{1}{4}\sum_{i=1}^N \mathbb{E}\left(\left\{\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{l}_i,\ldots,\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_S^{-1}\boldsymbol{l}_i\right\}^\top \left\{\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{l}_i,\ldots,\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_S^{-1}\boldsymbol{l}_i\right\}\right). \end{aligned} \quad (19)$$

By using $(\mathrm{dlog}|\boldsymbol{\Sigma}|)_l = \mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\mathrm{d}\boldsymbol{\Sigma}_l\right)$, the (k,l) component is given by:

$$\begin{aligned}(\mathcal{I}_3)_{k,l} &= \frac{N}{4}\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_k\right)\,\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\;\;\mathrm{d}\boldsymbol{\Sigma}_l\right) \\ &+ \frac{1}{4}\,\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_k\right)\,\sum_{i=1}^N \mathbb{E}\left(\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_l^{-1}\,\boldsymbol{l}_i\right) \\ &+ \frac{1}{4}\,\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_k\right)\,\sum_{i=1}^N \mathbb{E}\left(\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_l^{-1}\,\boldsymbol{l}_i\right) \\ &+ \frac{1}{4}\sum_{i=1}^N \mathbb{E}\left(\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_k^{-1}\,\boldsymbol{l}_i\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_l^{-1}\boldsymbol{l}_i\right). \end{aligned} \quad (20)$$

By definition, $\boldsymbol{\Sigma}$ is the covariance of the model (5) and:

$$\boldsymbol{\Sigma} = \mathbb{E}\left(\boldsymbol{l}_i\boldsymbol{l}_i^\top\right) \quad (21)$$

$$\frac{\partial\,\boldsymbol{\Sigma}(\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})\,)^{-1}}{\partial \delta_k}\bigg|_{\boldsymbol{\delta}=\mathbf{0}} = -\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_k\,\boldsymbol{\Sigma}^{-1} \quad (22)$$

then we have that,

$$\mathbb{E}\left(\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_k\boldsymbol{l}_i\right) = -\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_k\right). \quad (23)$$

It ensues, by substitution of (22) and (23) in (20),

$$(\mathcal{I}_3)_{k,l} = -\frac{1}{4}\sum_{i=1}^{N} \text{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_k\right)\,\text{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_l\right) + \frac{1}{4}\sum_{i=1}^{N}\mathbb{E}\left(\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_k^{-1}\,\boldsymbol{l}_i\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_l^{-1}\boldsymbol{l}_i\right). \quad (24)$$

Furthermore, by using the fact that $\boldsymbol{l}_i \sim \mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})$, we use the following identity [11, Appendix 15-C , pp. 565]:

$$\mathbb{E}\left(\boldsymbol{l}_i^\top\mathrm{d}\boldsymbol{\Sigma}_k^{-1}\,\boldsymbol{l}_i\,\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_l^{-1}\boldsymbol{l}_i)\right) = 2\,\text{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_k\,\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_l\right) + \text{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_k\right)\,\text{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_l\right), \quad (25)$$

which provides,

$$(\mathcal{I}_3)_{k,l} = \frac{N}{2}\,\text{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_k\,\boldsymbol{\Sigma}^{-1}\,\mathrm{d}\boldsymbol{\Sigma}_l\right). \quad (26)$$

IV. Closed-Form Expressions and Numerical Simulations

In this section, we derive the full Slepian-Bangs in a closed-form for the well-known Wahba's rotation model on $SO(3)$. In this framework, the uncertainty on the unobserved cloud points results in a covariance matrix of the observations that depends on the unknown rotation between the cloud points.

A. Formulation of the problem

The Wabha's problem consists in finding the unknown rotation $\mathbf{M} \in SO(3)$ connecting two 3D point clouds $\{\mathbf{z}_i\}_{i=1}^N$ and $\{\mathbf{p}_i\}_{i=1}^N$, expressed in two different frames. This can be modeled as,

$$\mathbf{z}_i = \mathbf{M}\,\mathbf{p}_i + \mathbf{n}_i \quad \forall i \in \{1,\ldots,N\} \;\; \mathbf{n}_i \sim \mathcal{N}(\mathbf{0},\mathbf{Q}). \quad (27)$$

In addition to the measurement noise of $\mathbf{z}_i$, the points $\{\mathbf{p}_i\}_{i=1}^N$ are also measured with some uncertainties. They can be modeled by the following Gaussian distribution with mean $\mathbf{p}_i^p$ and covariance matrix $\mathbf{Q}^p$,

$$p(\mathbf{p}_i) = \mathcal{N}(\mathbf{p}_i;\mathbf{p}_i^p,\mathbf{Q}^p). \quad (28)$$

Consequently, the distribution of $\mathbf{z}_i$ knowing $\mathbf{M}$ can be rewritten by using the conditional property and Gaussian distribution properties,

$$p(\mathbf{z}_i|\mathbf{M}) = \int p(\mathbf{z}_i|\mathbf{p}_i^p,\mathbf{M})p(\mathbf{p}_i^p)\mathrm{d}\mathbf{p}_i^p \quad (29)$$

$$=\mathcal{N}(\mathbf{z}_i;\mathbf{M}\mathbf{p}_i^p,\mathbf{M}\,\mathbf{Q}^p\,\mathbf{M}^\top+\mathbf{Q}). \quad (30)$$

Hence, we are faced with a problem where both mean and covariance depend on the unknown parameter $\mathbf{M}$. Then, the previous model can be rewritten on the LG $G' = \mathbb{R}^3$ with the compact CGD form:

$$\mathbf{Z}_i = \mathbf{H}_i(\mathbf{M})\;\mathrm{Exp}_{\mathbb{R}^3}^{\wedge}(\boldsymbol{\epsilon}_i)\;, \qquad \boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0},\boldsymbol{\Sigma}(\mathbf{M})), \quad (31)$$

with $\mathbf{H}_i(\mathbf{M}) \triangleq \begin{bmatrix}\mathbf{I} & \mathbf{M}\mathbf{p}_i^p\\ \mathbf{0} & 1\end{bmatrix}$, $\mathrm{Exp}_{\mathbb{R}^3}^{\wedge}(\boldsymbol{\epsilon}_i) \triangleq \begin{bmatrix}\mathbf{I} & \boldsymbol{\epsilon}_i\\ \mathbf{0} & 1\end{bmatrix}$ and $\boldsymbol{\Sigma}(\mathbf{M}) \triangleq \mathbf{M}\,\mathbf{Q}^p\,\mathbf{M}^\top+\mathbf{Q}$.

B. Full Slepian-Bangs formula computation

*Theorem IV-B.1 (**LG-F-SP for the Wabha's problem**): We consider the LG observation defined by equation (31). Furthermore, let us define $\{\mathbf{G}_l\}_{l=1}^3$ a basis of $\mathfrak{se}(3)$. The Slepian-Bangs formula is given by $\forall (k,l) \in \{1,2,3\}^2$:*

$$\mathcal{I} = \mathcal{I}_R + \mathcal{I}_\Sigma \quad (32)$$

$$(\mathcal{I}_R)_{k,l} = \sum_{i=1}^{N}(\mathbf{p}_i^p)^\top\,\mathbf{G}_k^\top\,\mathbf{M}^\top\boldsymbol{\Sigma}(\mathbf{M})^{-1}\mathbf{M}\,\mathbf{G}_l\,\mathbf{p}_i^p \quad (33)$$

$$(\mathcal{I}_\Sigma)_{k,l} = \frac{N}{2}\,\text{tr}\left(\boldsymbol{\Sigma}(\mathbf{M})^{-1}\,\left(\mathbf{M}\,\mathbf{G}_k\,\mathbf{Q}^p\,\mathbf{M}^\top+\mathbf{M}\,\mathbf{Q}^p\,\mathbf{G}_k^\top\,\mathbf{M}^\top\right)\boldsymbol{\Sigma}(\mathbf{M})^{-1}\,\left(\mathbf{M}\,\mathbf{G}_l\,\mathbf{Q}^p\,\mathbf{M}^\top+\mathbf{M}\,\mathbf{Q}^p\,\mathbf{G}_l^\top\,\mathbf{M}^\top\right)\right) \quad (34)$$

<u>*Demonstration*</u>
We start by using the formula (6):

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_2^\top + \mathcal{I}_3 \quad (35)$$

- First, we observe that

$$\left[\mathcal{L}^R_{\mathbf{H}_i(\mathbf{M})}\right]_l = \left(\left.\frac{\partial \mathbf{H}_i(\mathbf{M}\mathrm{Exp}_{SO(3)}^{\wedge}(\boldsymbol{\delta})\,)}{\partial\boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}\right)_l \quad (36)$$

$$= \mathbf{M}\left.\frac{\partial \mathrm{Exp}_{SO(3)}^{\wedge}(\boldsymbol{\delta})}{\partial \delta_l}\right|_{\boldsymbol{\delta}=\mathbf{0}}\mathbf{p}_i^p = \mathbf{M}\mathbf{G}_l\mathbf{p}_i^p \quad (37)$$

  then, we deduce that $\mathcal{I}_1 = \mathcal{I}_R$
- Second, we remark that $\mathcal{I}_2$ is null. Indeed, the LG of the observations is $\mathbb{R}^3$, hence commutative. It implies that:

$$\mathcal{I}_2 = \boldsymbol{\Sigma}(\mathbf{M})^{-1}\mathbb{E}\left(\{\boldsymbol{l}_i\boldsymbol{l}_i^\top\mathrm{d}\boldsymbol{\Sigma}_1\boldsymbol{l}_i,\ldots,\boldsymbol{l}_i\boldsymbol{l}_i^\top\,\mathrm{d}\boldsymbol{\Sigma}_S\,\boldsymbol{l}_i\}\right) \quad (38)$$

  As $\boldsymbol{l}_i$ is a centered Gaussian vector, $\mathbb{E}\left(\boldsymbol{l}_i\,\boldsymbol{l}_i^\top\mathrm{d}\boldsymbol{\Sigma}_S\,\boldsymbol{l}_i\right) = \mathbf{0}$.
- Third, the term $\mathcal{I}_3$ can be detailed with (20) by computing the term $\mathrm{d}\boldsymbol{\Sigma}_k$.

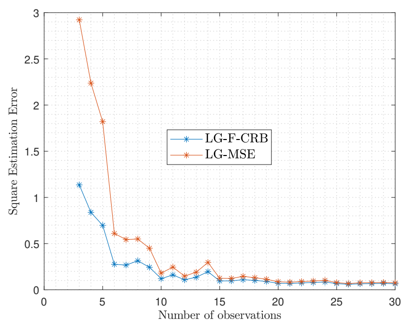
$$\mathrm{d}\boldsymbol{\Sigma}_k = \left.\frac{\partial\boldsymbol{\Sigma}(\mathbf{M}\mathrm{Exp}_{SO(3)}^{\wedge}(\boldsymbol{\delta})\,)}{\partial\delta_k}\right|_{\boldsymbol{\delta}=\mathbf{0}} \quad (39)$$

$$= \left.\frac{\partial\left(\mathbf{M}\,\mathrm{Exp}_{SO(3)}^{\wedge}(\boldsymbol{\delta})\;\;\mathbf{Q}^p\mathrm{Exp}_{SO(3)}^{\wedge}(\boldsymbol{\delta})\;^\top\mathbf{M}^\top\right)}{\partial\delta_k}\right|_{\boldsymbol{\delta}=\mathbf{0}} \quad (40)$$

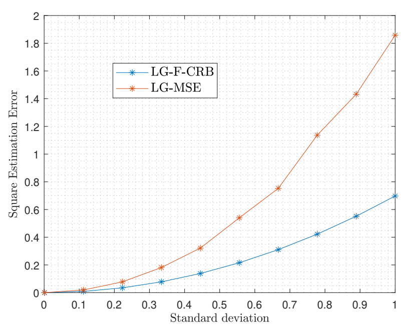
$$= \mathbf{M}\,\mathbf{G}_k\,\mathbf{Q}^p\,\mathbf{M}^\top + \mathbf{M}\,\mathbf{Q}^p\,\mathbf{G}_k^\top\,\mathbf{M}^\top \quad (41)$$

C. Simulation results

In this section, we propose to test and validate the proposed Slepian-Bangs formula provided by equation (32). To achieve that, we first simulate observations according to the formula with arbitrary values of $\mathbf{p}_i^p$, $\mathbf{Q} = \sigma^2\mathbf{I}_3$ and $\mathbf{Q}^p = \sigma_p^2\mathbf{I}_3$. Then, we compare the inverse of the LG-F-SP (32), which yields the LG-F-CRB with the empirical LG-MSE given by $\frac{1}{N_r}\sum_{nr=1}^{N_r}\|\mathrm{Log}_G^{\vee}\left(\mathbf{M}^{-1}\widehat{\mathbf{M}}^{(nr)}\right)\|^2$, where $\widehat{\mathbf{M}}^{(nr)}$ is the n_r realization of the maximum likelihood estimator of the model (31) minimizing $\sum_{i=1}^{N}\|\mathbf{z}_i-\mathbf{M}\mathbf{p}_i^p\|^2_{\boldsymbol{\Sigma}^{-1}(\mathbf{M})}+N\log|\boldsymbol{\Sigma}(\mathbf{M})|$. The latter is obtained iteratively with a Gauss-Newton algorithm, where at each iteration l, $\mathbf{M}^{(l)}$ is updated by minimizing $\sum_{i=1}^{N} \|\mathbf{z}_i - \mathbf{M}\mathbf{p}_i^p\|^2_{\mathbf{\Sigma}^{-1}(\mathbf{M}^{(l)})}$. The unknown $\mathbf{\Sigma}(\mathbf{M}^{(l)})$ is updated by $\frac{1}{N-1}\sum_{i=1}^{N}(\mathbf{z}_i - \mathbf{M}^{(l)}\mathbf{p}_i^p)(\mathbf{z}_i - \mathbf{M}^{(l)}\mathbf{p}_i^p)^\top$. In Figures 1a and 1b, we draw respectively both LG-MSE and LG-CRB, w.r.t. the number of observations for $\sigma^2 = 0.01^2$ and $\sigma_p^2 = 1^2$, and w.r.t. varying values of the standard deviation σ for a fixed $N = 5$.



(a) LG-CRB and LG-MSE w.r.t the number of observations N.



(b) LG-CRB and LG-MSE w.r.t varying values of the measurement noise σ^2.

We observe the consistency of the LG-CRB with regards to the LG-MSE, particularly its asymptotic behavior. Specifically, in Fig. 1a, the LG-MSE aligns with the LG-F-CRB when the number of observations increases. Moreover in Fig. 1b the LG-F-CRB and the LG-MSE align when measurement noise variance is low. If the latter increases, the LG-MSE drifts away due to bias. This behaviour, in line with the Euclidean case, validates the proposed LG-F-SP.

V. Conclusions and Perspectives

In this article, we derived a full Slepian-Bangs formula on LGs for Gaussian distributions on LGs, where the unknown parameter is embedded in both LG-mean and LG-covariance. Future directions of this research will involve extending the Slepian-Bangs formula to LG models incorporating non-Gaussian noise in the Lie algebra. Specifically, we may focus on modeling observations with elliptical noise distributions to characterize heavy-tailed behaviours.

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